

# ON GENERALIZED POWERS-STØRMER'S INEQUALITY

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ABSTRACT. A generalization of Powers-Størmer's inequality for operator monotone functions on  $[0, +\infty)$  and for positive linear functional on general  $C^*$ -algebras will be proved. It also will be shown that the generalized Powers-Størmer inequality characterizes the tracial functionals on  $C^*$ -algebras.

## 1. INTRODUCTION

Powers-Størmer's inequality (see, for example, [12]) asserts that for  $s \in [0, 1]$  the following inequality

$$(1) \quad 2 \operatorname{Tr}(A^s B^{1-s}) \geq \operatorname{Tr}(A + B - |A - B|)$$

holds for any pair of positive matrices  $A, B$ . This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory [1]. This inequality was first proven in [1], using an integral representation of the function  $t^s$ . After that, M. Ozawa gave a much simpler proof for the same inequality, using fact that for  $s \in [0, 1]$  function  $f(t) = t^s$  ( $t \in [0, +\infty)$ ) is an operator monotone. Recently, Y. Ogata in [10] extended this inequality to standard von Neumann algebras. The motivation of this paper is that if the function  $f(t) = t^s$  is replaced by another operator monotone function (this class is intensively studied, see [7][11]), then  $\operatorname{Tr}(A + B - |A - B|)$  may get smaller upper bound that is used in quantum hypothesis testing. Based on M. Ozawa's proof we formulate Powers-Størmer's inequality for an arbitrary operator monotone function on  $[0, +\infty)$  in the context of general  $C^*$ -algebras.

Finally, we will show that the Powers-Størmer's inequality characterizes the trace property for a normal linear positive functional on a von Neumann algebras and for a linear positive functional on a  $C^*$ -algebra.

Recall that a positive linear functional  $\varphi$  on a von Neumann algebra  $\mathcal{M}$  is said to be *normal* if  $\varphi(\sup A_i) = \sup \varphi(A_i)$  for every bounded increasing net  $A_i$  of positive elements in  $\mathcal{M}$ . A linear functional  $\varphi$  on a  $C^*$ -algebra  $\mathcal{A}$  is said to be *tracial* if  $\varphi(AB) = \varphi(BA)$  for all  $A, B \in \mathcal{A}$ .

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For all other notions used in the paper, we refer the reader to the monograph [8].

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## 2. MAIN RESULTS

Let  $n \in \mathbf{N}$  and  $M_n$  be the algebra of  $n \times n$  matrices. Let  $I$  be an interval in  $\mathbf{R}$  and  $f: I \rightarrow \mathbf{R}$  be a continuous function. We call a function  $f$  matrix monotone of order  $n$  or  $n$ -monotone in short whenever the inequality

$$A \leq B \implies f(A) \leq f(B)$$

for an arbitrary selfadjoint matrices  $A, B \in M_n$  such that  $A \leq B$  and all eigenvalues of  $A$  and  $B$  are contained in  $I$ .

Let  $H$  be a separable infinite dimensional Hilbert space and  $B(H)$  be the set of all bounded linear operators on  $H$ . We call a function  $f$  operator monotone whenever the inequality

$$A \leq B \implies f(A) \leq f(B)$$

for an arbitrary selfadjoint matrices  $A, B \in B(H)$  such that  $A \leq B$  and all eigenvalues of  $A$  and  $B$  are contained in  $I$ .

We denote the spaces of operator monotone functions by  $P_\infty(I)$ . The spaces for  $n$ -monotone functions are written as  $P_n(I)$ . We have then

$$P_1(I) \supseteq \cdots \supseteq P_{n-1}(I) \supseteq P_n(I) \supseteq P_{n+1}(I) \supseteq \cdots \supseteq P_\infty(I).$$

Here we note that  $\bigcap_{n=1}^\infty P_n(I) = P_\infty(I)$  and each inclusion is proper [7][11].

The following result is well-known. For example see the proof in [5, Theorem 2.5].

**Lemma 2.1.** *Let  $f$  be a strictly positive, continuous function on  $[0, \infty)$ . If the function  $f$  is  $2n$ -monotone, then for any positive semidefinite  $A$  and a contraction  $C$  in  $M_n$  we have*

$$C^* f(A) C \leq f(C^* A C).$$

**Lemma 2.2.** *Let  $f$  be a continuous function on  $(0, \infty)$  such that  $0 \notin f((0, \infty))$ . Then,  $f$  is  $n$ -monotone if and only if the function  $-\frac{1}{f(t)}$  is  $n$ -monotone.*

*Proof.* For any  $t_1, t_2, \dots, t_n \in (0, \infty)$  we have

$$\begin{aligned} \frac{\frac{1}{f(t_i)} - \frac{1}{f(t_j)}}{t_i - t_j} &= \frac{\frac{f(t_j) - f(t_i)}{f(t_i)f(t_j)}}{t_i - t_j} \\ &= -\frac{1}{f(t_i)f(t_j)} \frac{f(t_i) - f(t_j)}{t_i - t_j}. \end{aligned}$$

Since  $f$  is  $n$ -monotone, the matrix  $[\frac{f(t_i) - f(t_j)}{t_i - t_j}]$  is positive semidefinite by [9], hence, we have

$$\begin{aligned} \left[ \frac{\left(-\frac{1}{f(t_i)}\right) - \left(-\frac{1}{f(t_j)}\right)}{t_i - t_j} \right] &= - \left[ \frac{\frac{1}{f(t_i)} - \frac{1}{f(t_j)}}{t_i - t_j} \right] \\ &= - \left( - \left[ \frac{1}{f(t_i)f(t_j)} \frac{f(t_i) - f(t_j)}{t_i - t_j} \right] \right) \\ &= \left[ \frac{1}{f(t_i)f(t_j)} \right] \circ \left[ \frac{f(t_i) - f(t_j)}{t_i - t_j} \right] \\ &\geq 0, \end{aligned}$$

where  $\circ$  means the Hadamard product.

Therefore, the function  $-\frac{1}{f(t)}$  is  $n$ -monotone by [9].

Conversely, if  $-\frac{1}{f}$  is  $n$ -monotone, we have

$$\begin{aligned} \left[ \frac{f(t_i) - f(t_j)}{t_i - t_j} \right] &= [f(t_i)f(t_j)] \circ \left[ \frac{\left(-\frac{1}{f(t_i)}\right) - \left(-\frac{1}{f(t_j)}\right)}{t_i - t_j} \right] \\ &\geq 0, \end{aligned}$$

hence  $f$  is  $n$ -monotone. □

**Proposition 2.1.** *Let  $f$  be a strictly positive, continuous function on  $[0, \infty)$ . If  $f$  is  $2n$ -monotone, the function  $g(t) = \frac{t}{f(t)}$  is  $n$ -monotone on  $[0, \infty)$ .*

*Proof.* Let  $A, B$  be positive matrixes in  $M_n$  such that  $0 < A \leq B$ .

Let  $C = B^{-\frac{1}{2}}A^{\frac{1}{2}}$ . Then  $\|C\| \leq 1$ . Since  $f$  is  $2n$ -monotone,  $-f$  satisfies the Jensen type inequality from Lemma 2.1, that is,

$$\begin{aligned} -f(A) &= -f(C^*BC) \leq -C^*f(B)C \\ -f(A) &\leq -A^{\frac{1}{2}}B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}A^{\frac{1}{2}} \\ -A^{-\frac{1}{2}}f(A)A^{-\frac{1}{2}} &\leq -B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}} \\ -A^{-1}f(A) &\leq -B^{-1}f(B) \end{aligned}$$

Hence, the function  $-\frac{f(t)}{t}$  is  $n$ -monotone. Therefore, from Lemm 2.2 we conclude that

$$-\frac{1}{-\frac{f(t)}{t}} = \frac{t}{f(t)}$$

is  $n$ -monotone.  $\square$

*Remark 1.* The condition of  $2n$ -monotonicity of  $f$  is needed to guarantee the  $n$ -monotonicity of  $g$ . Indeed, it is well-known that  $t^3$  is monotone, but not 2-monotone. In this case the function  $g(t) = \frac{t}{t^3} = \frac{1}{t^2}$  is obviously not 1-monotone.

**Proposition 2.2.** *Let  $h: [0, \infty) \rightarrow [0, \infty)$  be a Borel function such that  $h$  is a continuous,  $n$ -monotone on  $(0, \infty)$ , and  $h(0) = 0$ . Then for any  $A, B \in M_n^+$  with  $A \leq B$  we have*

$$h(A) \leq h(B).$$

*Proof.* Let  $B = \sum_s \mu_s q_s$  be a spectral decomposition. Set  $1 - q$  as a projection on  $\text{Ker}(B)$ . Then  $B = Bq = qB = \sum_{s'} \mu_{s'} q_{s'}$  and  $q = \sum_{s'} q_{s'}$ .

Similarly, let  $A = \sum_t \lambda_t p_t$  be a spectral projection and  $(1 - p)$  be a projection on  $\text{Ker}(A)$ . Since  $A \leq B$ ,  $p \leq q$  and  $A = \sum_{t'} \lambda_{t'} p_{t'}$  and  $p = \sum_{t'} p_{t'}$ . Note that since  $h(0) = 0$ , by the function calculus we have  $h(A) = \sum_{t'} h(\lambda_{t'}) p_{t'}$  and  $h(B) = \sum_{s'} h(\mu_{s'}) q_{s'}$ .

For any  $\varepsilon > 0$  since

$$\begin{aligned} 0 &< \sum_{t'} \lambda_{t'} p_{t'} + \varepsilon 1 \\ &\leq \sum_{s'} \mu_{s'} q_{s'} + \varepsilon 1 \end{aligned}$$

and  $h$  is  $n$ -monotone on  $(0, \infty)$ , we have

$$h\left(\sum_{t'} (\lambda_{t'} + \varepsilon) p_{t'} + \varepsilon(1 - p)\right) \leq h\left(\sum_{s'} (\mu_{s'} + \varepsilon) q_{s'} + \varepsilon(1 - q)\right).$$

Since

$$\begin{aligned} \sum_{t'} h(\lambda_{t'} + \varepsilon) p_{t'} + h(\varepsilon)(1 - p) &= h\left(\sum_{t'} (\lambda_{t'} + \varepsilon) p_{t'} + \varepsilon(1 - p)\right) \\ &\leq h\left(\sum_{s'} (\mu_{s'} + \varepsilon) q_{s'} + \varepsilon(1 - q)\right) \\ &= \sum_{s'} h(\mu_{s'} + \varepsilon) q_{s'} + h(\varepsilon)(1 - q) \end{aligned}$$

and  $p \leq q$ , it follows that

$$\begin{aligned} \sum_{t'} h(\lambda_{t'} + \varepsilon) p_{t'} &\leq \sum_{t'} h(\lambda_{t'} + \varepsilon) p_{t'} + h(\varepsilon) q(1 - p) q \\ &\leq \sum_{s'} h(\mu_{s'} + \varepsilon) q_{s'}. \end{aligned}$$

Therefore, since  $h$  is continuous on  $(0, \infty)$ , as  $\varepsilon \rightarrow 0$  we have

$$\begin{aligned} h(A) &= \sum_{t'} h(\lambda_{t'}) p_{t'} \\ &\leq \sum_{s'} h(\mu_{s'}) q_{s'} \\ &= h(B). \end{aligned}$$

□

**Corollary 2.1.** *Let  $f$  be a  $2n$ -monotone, continuous function on  $[0, \infty)$  such that  $f((0, \infty)) \subset (0, \infty)$ , and let  $g$  be a Borel function on  $[0, \infty)$  defined by  $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$ . Then for any pair of positive matrices  $A, B \in M_n$  with  $A \leq B$ ,  $g(A) \leq g(B)$ .*

*Proof.* Since  $f$  is  $2n$ -monotone, continuous function on  $[0, \infty)$  such that  $f((0, \infty)) \subset (0, \infty)$ , from Proposition 2.1  $g$  is  $n$ -monotone on  $(0, \infty)$ .

Hence, since  $g$  is a Borel function on  $[0, \infty)$  with  $g(0) = 0$ , from Proposition 2.2 it follows that  $g(A) \leq g(B)$ . □

**Theorem 2.1.** *Let  $\text{Tr}$  be a canonical trace on  $M_n$  and  $f$  be a  $2n$ -monotone function on  $[0, \infty)$  such that  $f((0, \infty)) \subset (0, \infty)$ . Then for any pair of positive matrices  $A, B \in M_n$*

$$(2) \quad \text{Tr}(A) + \text{Tr}(B) - \text{Tr}(|A - B|) \leq 2 \text{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

$$\text{where } g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}.$$

*Proof.* Let  $A, B$  be any positive matrices in  $M_n$ .

For operator  $(A - B)$  let us denote by  $P = (A - B)^+$  and  $Q = (A - B)^-$  its positive and negative part, respectively. Then we have

$$(3) \quad A - B = P - Q \quad \text{and} \quad |A - B| = P + Q,$$

from that it follows that

$$(4) \quad A + Q = B + P.$$

On account of (4) the inequality (2) is equivalent to the following

$$\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) \leq \mathrm{Tr}(P).$$

Since  $B + P \geq B \geq 0$  and  $B + P = A + Q \geq A \geq 0$ , we have  $g(A) \leq g(B + P)$  by Corollary 2.1 and

$$\begin{aligned} & \mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) \\ &= \mathrm{Tr}(f(A)^{\frac{1}{2}}g(A)f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) \\ &\leq \mathrm{Tr}(f(A)^{\frac{1}{2}}g(B + P)f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) \\ &= \mathrm{Tr}(f(A)^{\frac{1}{2}}(g(B + P) - g(B))f(A)^{\frac{1}{2}}) \\ &\leq \mathrm{Tr}(f(B + P)^{\frac{1}{2}}(g(B + P) - g(B))f(B + P)^{\frac{1}{2}}) \\ &= \mathrm{Tr}(f(B + P)^{\frac{1}{2}}g(B + P)f(B + P)^{\frac{1}{2}}) \\ &\quad - \mathrm{Tr}(f(B + P)^{\frac{1}{2}}g(B)f(B + P)^{\frac{1}{2}}) \\ &\leq \mathrm{Tr}(B + P) - \mathrm{Tr}(f(B)^{\frac{1}{2}}g(B)f(B)^{\frac{1}{2}}) \\ &= \mathrm{Tr}(B + P) - \mathrm{Tr}(B) \\ &= \mathrm{Tr}(P). \end{aligned}$$

Hence, we have the conclusion. □

*Remark 2.* (i) When given positive matrices  $A, B$  in  $M_n$  satisfies the condition  $A \leq B$ , the inequality (2) becomes

$$\mathrm{Tr}(A) \leq \mathrm{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}).$$

(ii) As pointed in Proposition 2.1, 2-monotonicity of  $f$  is needed to guarantee the inequality (2). Indeed, let  $f(t) = t^3$  and  $n = 1$ . Then, for any  $a, b \in (0, \infty)$ , the inequality (2) would imply

$$a \leq f(a)^{\frac{1}{2}}g(b)f(a)^{\frac{1}{2}},$$

that is,

$$\frac{a}{f(a)} \leq \frac{b}{f(b)}.$$

Since  $\frac{t}{f(t)}$  is, however, not 1-monotone, the latter inequality is impossible.

As an application we get Powers-Størmer's inequality.

**Corollary 2.2.** [1, Theorem 1] *Let  $A$  and  $B$  be positive matrices, then for all  $s \in [0, 1]$*

$$\mathrm{Tr}(A + B - |A - B|) \leq \mathrm{Tr}(A^s B^{1-s}).$$

*Proof.* Let  $f(t) = t^s$  ( $s \in [0, 1]$ ). Then  $f$  is operator monotone with  $f(0, \infty) \subset (0, \infty)$  and  $g(t) = t^{1-s}$ . Hence, we have the conclusion from Theorem 2.1.  $\square$

Since any  $C^*$ -algebra can be realized as a closed selfadjoint  $*$ -algebra of  $B(H)$  for some Hilbert space  $H$ . We can generalize Theorem 2.1 in the framework of  $C^*$ -algebras.

**Theorem 2.2.** *Let  $\tau$  be a tracial functional on a  $C^*$ -algebra  $\mathcal{A}$ ,  $f$  be a strictly positive, operator monotone function on  $[0, \infty)$ . Then for any pair of positive elements  $A, B \in \mathcal{A}$*

$$(5) \quad \tau(A) + \tau(B) - \tau(|A - B|) \leq 2\tau(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}),$$

where  $g(t) = tf(t)^{-1}$ .

*Proof.* Since the function  $\frac{t}{f(t)}$  is operator monotone on  $(0, \infty)$  by [5, Corollary 6], we can get the conclusion through the same steps in the proof of Theorem 2.1.  $\square$

*Remark 3.* For matrices  $A, B \in M_n^+$  let us denote

$$(6) \quad Q(A, B) = \min_{s \in [0, 1]} \text{Tr}(A^{(1-s)/2} B^s A^{(1-s)/2})$$

and

$$(7) \quad Q_{\mathcal{F}_{2n}}(A, B) = \inf_{f \in \mathcal{F}_{2n}} \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}),$$

where  $\mathcal{F}_{2n}$  is the set of all  $2n$ -monotone functions on  $[0, +\infty)$  satisfy condition of the Theorem 2.1 and  $g(t) = tf(t)^{-1}$  ( $t \in [0, +\infty)$ ).

Note that the function  $f(t) = t^s$  ( $t \in [0, +\infty)$ ) satisfies the conditions of Theorem 2.1. Since the class of  $2n$ -monotone functions is large enough [11], we know that  $Q_{\mathcal{F}_{2n}}(A, B) \leq Q(A, B)$ . Hence, we hope on finding another  $2n$ -monotone function  $f$  on  $[0, +\infty)$  such that

$$(8) \quad \text{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) < Q(A, B).$$

If we can find such a function, then we can refine the quantum Chernoff bound used in quantum hypothesis testing [1].

### 3. CHARACTERIZATIONS OF THE TRACE PROPERTY

In this section the generalized Powers-Størmer inequality in the previous section implies the trace property for a positive linear functional on operator algebras.

**Lemma 3.1.** *Let  $\varphi$  be a positive linear functional on  $M_n$  and  $f$  be a continuous function on  $[0, \infty)$  such that  $f(0) = 0$  and  $f((0, \infty)) \subset (0, \infty)$ . If the following inequality*

$$(9) \quad \varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})$$

holds true for all  $A, B \in M_n^+$ , then  $\varphi$  should be a positive scalar multiple of the canonical trace  $\text{Tr}$  on  $M_n$ , where  $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$ .

*Proof.* As is well known, every positive linear functional  $\varphi$  on  $M_n$  can be represented in the form  $\varphi(\cdot) = \text{Tr}(S_\varphi \cdot)$  for some  $S_\varphi \in M_n^+$ . It is easily seen that without loss of generality we can assume that  $S_\varphi = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ , and we have to prove that  $\alpha_i = \alpha_j$  for all  $i, j = 1, \dots, n$ . Clearly, it is sufficient to prove that  $\alpha_1 = \alpha_2$ . By assumption, the inequality (9) holds true, in particular, for any positive matrices  $X = [x_{ij}]_{i,j=1}^n, Y = [y_{ij}]_{i,j=1}^n$  from  $M_n^+$  such that  $0 = x_{ij} = y_{ij}$  if  $3 \leq i \leq n$  or  $3 \leq j \leq n$ . Thus, it suffices to consider the case  $n = 2$ . Assume that  $S_\varphi = \text{diag}(d, 1)$  ( $d \in [0, 1]$ ) and  $\varphi(D) = \text{Tr}(S_\varphi D), \forall D \in M_2$ . We show that  $d = 1$ . For arbitrary positive numbers  $\lambda, \mu$  such that  $\lambda < \mu$  we consider the following matrices

$$A = \begin{pmatrix} \lambda & \sqrt{\lambda\mu} \\ \sqrt{\lambda\mu} & \mu \end{pmatrix}$$

and

$$B = \begin{pmatrix} \lambda & -\sqrt{\lambda\mu} \\ -\sqrt{\lambda\mu} & \mu \end{pmatrix}.$$

It is clear that these are positive scalar multiple of projections of rank one. In addition,

$$f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}} = \left( \frac{\mu - \lambda}{\mu + \lambda} \right)^2 A.$$

We have

$$\begin{aligned} 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) &= 2 \left( \frac{\mu - \lambda}{\mu + \lambda} \right)^2 \text{Tr}(S_\varphi A) \\ &= 2 \left( \frac{\mu - \lambda}{\mu + \lambda} \right)^2 (d\lambda + \mu). \end{aligned}$$

By direct calculation,

$$|A - B| = \begin{pmatrix} 2\sqrt{\lambda\mu} & 0 \\ 0 & 2\sqrt{\lambda\mu} \end{pmatrix}.$$

Consequently,

$$\varphi(A + B) - \varphi(|A - B|) = d(2\lambda - 2\sqrt{\lambda\mu}) + 2\mu - 2\sqrt{\lambda\mu}.$$

Then the inequality (9) becomes

$$\left( \frac{\mu - \lambda}{\mu + \lambda} \right)^2 (d\lambda + \mu) \geq d(\lambda - \sqrt{\lambda\mu}) + \mu - \sqrt{\lambda\mu}.$$

Dividing two side by  $\sqrt{\lambda}(\sqrt{\mu} - \sqrt{\lambda})$ , we get

$$d + \frac{(\sqrt{\mu} - \sqrt{\lambda})(\sqrt{\mu} + \sqrt{\lambda})^2}{\sqrt{\lambda}(\mu + \lambda)^2} (d\lambda + \mu) \geq \sqrt{\frac{\mu}{\lambda}}.$$



Tending  $\lambda$  to  $\mu$  from the left we obtain

$$d \geq 1.$$

Since  $d \in [0, 1]$ ,  $d = 1$ . This means that  $\varphi$  is a positive scalar multiple of the canonical trace  $\text{Tr}$  on  $M_n$ .  $\square$

*Remark 4.* Let  $\varphi$  be a positive linear functional on  $M_n$  and  $s \in [0, 1]$ . From Lemma 3.1 it is clear that if the following inequality

$$(10) \quad \varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(A^{\frac{1-s}{2}} B^s A^{\frac{1-s}{2}})$$

holds true for any  $A, B \in M_n^+$ , then  $\varphi$  is a tracial. In particular, when  $s = 0$  the following inequality characterizes the trace property

$$(11) \quad \varphi(B) - \varphi(A) \leq \varphi(|A - B|) \quad (A, B \in M_n^+).$$

**Corollary 3.1** ([13]). *Let  $\varphi$  be a positive linear functional on  $M_n$  and the following inequality*

$$(12) \quad \varphi(|A + B|) \leq \varphi(|A|) + \varphi(|B|)$$

*holds true for any self-adjoint matrices  $A, B \in M_n$ . Then  $\varphi$  is a tracial.*

*Proof.* From the assumption, we have

$$\varphi(|B - A|) \geq \varphi(|B|) - \varphi(|A|)$$

for any pair of self-adjoint matrices  $A, B$  in  $M_n$ . Moreover, for any pair of positive matrices  $A, B \in M_n$  we have

$$\varphi(|B - A|) \geq \varphi(B) - \varphi(A).$$

On account of Remark 4, it follows that  $\varphi$  should be a tracial.  $\square$

**Corollary 3.2** ([4]). *Let  $\varphi$  be a positive linear functional on  $M_n$  and the following inequality*

$$(13) \quad |\varphi(A)| \leq \varphi(|A|)$$

*holds true for any self-adjoint matrix  $A \in M_n$ . Then  $\varphi$  is a tracial.*

*Proof.* Let  $A, B \in M_n$  be arbitrary positive matrices. Then  $C = B - A$  is a self-adjoint matrix. Since  $A, B \geq 0$ , the values  $\varphi(A)$  and  $\varphi(B)$  are real. From the assumption, we have

$$\varphi(B) - \varphi(A) \leq |\varphi(B) - \varphi(A)| = |\varphi(B - A)| \leq \varphi(|B - A|).$$

On account of Remark 4, it follows that  $\varphi$  should be a tracial.  $\square$

By analogy with a number of other similar cases (see [4] or [14]), the proof for the trace property of a positive normal functional satisfying the inequality (9) on a von Neumann algebra can be reduced to the case of the algebra  $M_2$  of all matrices of order  $2 \times 2$ . But for self-contained we will give a sketch of its proof.

**Theorem 3.1.** *Let  $\varphi$  be a positive normal linear functional on a von Neumann algebra  $\mathcal{M}$  and  $f$  be a continuous function on  $[0, \infty)$  such that  $f(0) = 0$  and  $f((0, \infty)) \subset (0, \infty)$ . If the following inequality*

$$(14) \quad \varphi(A) + \varphi(B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})$$

*holds true for any pair  $A, B \in \mathcal{M}^+$ , then  $\varphi$  is a trace, where  $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$ .*

*Proof.* Let  $P_1, P_2$  be a pair of nonzero mutually orthogonal equivalent projections in  $\mathcal{M}$ , that means  $P_1 = V^*V$  and  $P_2 = VV^*$  for some nonzero partial isometry  $V \in \mathcal{M}$ . Consider the  $*$ -algebra  $\mathcal{N}$  in  $(P_1 + P_2)\mathcal{M}(P_1 + P_2)$  generated by the partial isometry  $V$ . Then  $\mathcal{N}$  is isomorphic to  $M_2$  and inequality (14) still holds true for the operators in  $\mathcal{N}$  and for the restriction of the functional  $\varphi$  to  $\mathcal{N}$ . According to Lemma 3.1, this restriction is a tracial functional on  $\mathcal{N}$ , and hence  $\varphi(P_1) = \varphi(P_2)$ . By [8, Vol2, Proposition 8.1.1] it follows that  $\varphi$  is a trace.  $\square$

**Corollary 3.3.** *Let  $\varphi$  be a positive linear functional on a  $C^*$ -algebra  $\mathcal{A}$  and  $f$  be a continuous function on  $[0, \infty)$  such that  $f(0) = 0$  and  $f((0, \infty)) \subset (0, \infty)$ . If the following inequality*

$$(15) \quad \varphi(A) + \varphi(B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})$$

*holds true for any pair  $A, B \in \mathcal{A}^+$ , then  $\varphi$  is a tracial functional, where  $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$ .*

*Proof.* Let  $\pi$  be the universal representation of  $C^*$ -algebra  $\mathcal{A}$  and  $\mathcal{M} = \pi(\mathcal{A})''$ . Let  $\hat{\varphi}$  be the positive normal functional on  $\mathcal{M}$  such that  $\hat{\varphi}(\pi(A)) = \varphi(A)$  for  $A \in \mathcal{A}$ . By the Kaplansky density theorem, for any pair  $\hat{A}, \hat{B} \in \mathcal{M}^+$  there exist bounded nets  $\{A_\alpha\}$  and  $\{B_\alpha\}$  in  $\mathcal{A}^+$  such that  $\pi(A_\alpha) \rightarrow \hat{A}$  and  $\pi(B_\alpha) \rightarrow \hat{B}$  in the strong operator topology. Using (15) and the continuity of the corresponding operations in the strong operator topology, we have

$$\hat{\varphi}(\hat{A}) + \hat{\varphi}(\hat{B}) - \hat{\varphi}(|\hat{A} - \hat{B}|) \leq 2\hat{\varphi}(f(\hat{A})^{\frac{1}{2}}g(\hat{B})f(\hat{A})^{\frac{1}{2}}).$$

By Theorem 3.1,  $\hat{\varphi}$  is a tracial functional  $\mathcal{M}$ , and hence  $\varphi$  is a tracial functional on  $\mathcal{A}$ .  $\square$

*Remark 5.* Let  $\mathcal{A}$  be a von Neumann algebra and  $\varphi$  be a positive linear functional on  $\mathcal{A}$ . The set  $P(\mathcal{A})$  of all orthogonal projections from  $\mathcal{A}$  is enough as a testing space for some inequality to characterize the trace property of  $\varphi$  (see [3]). But, in the case of the inequality (14) the set  $P(\mathcal{A})$  is not enough as a testing set.

Indeed, let  $p, q$  be arbitrary orthogonal projections from a von Neumann algebra  $\mathcal{M}$ . Since  $q \geq p \wedge q$  it follows that  $pqp \geq p(p \wedge q)p = p \wedge q$ . So  $pqp \geq p \wedge q$  holds for any pair of projections. From that it follows

$$\varphi(p + q - |p - q|) = 2\varphi(p \wedge q) \leq 2\varphi(pqp) = 2\varphi(f(p)^{\frac{1}{2}}g(q)f(p)^{\frac{1}{2}})$$

whenever  $\varphi$  is an arbitrary positive linear functional on  $\mathcal{M}$ .

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